

Classifying Trees by Chromatic Symmetric Functions of Trees

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Abstract

This thesis approaches the problem of classifying and distinguishing trees by their chromatic symmetric functions with the limited number of colors. The first part of the thesis is on the bicoloring of trees with n vertices. Every tree can be properly colored using two colors, with one color used c_1 times and the other color used $n - c_1$ times. Using polynomials in two variables, we partition the set of trees according to the number of times each color is used. The use of minimal units, which generate every tree that belongs to each block in the partition, allows us to define formulas and bijective relationships to count the size of each block in the partition. Chromatic symmetric functions using two colors do not distinguish trees with n vertices. Thus, we finally approach the problem of distinguishing trees by their chromatic symmetric functions and how many colors are needed. I show that, using chromatic symmetric functions with three colors, we can distinguish bicolorable trees with the fewer color used twice.

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Chapter 1

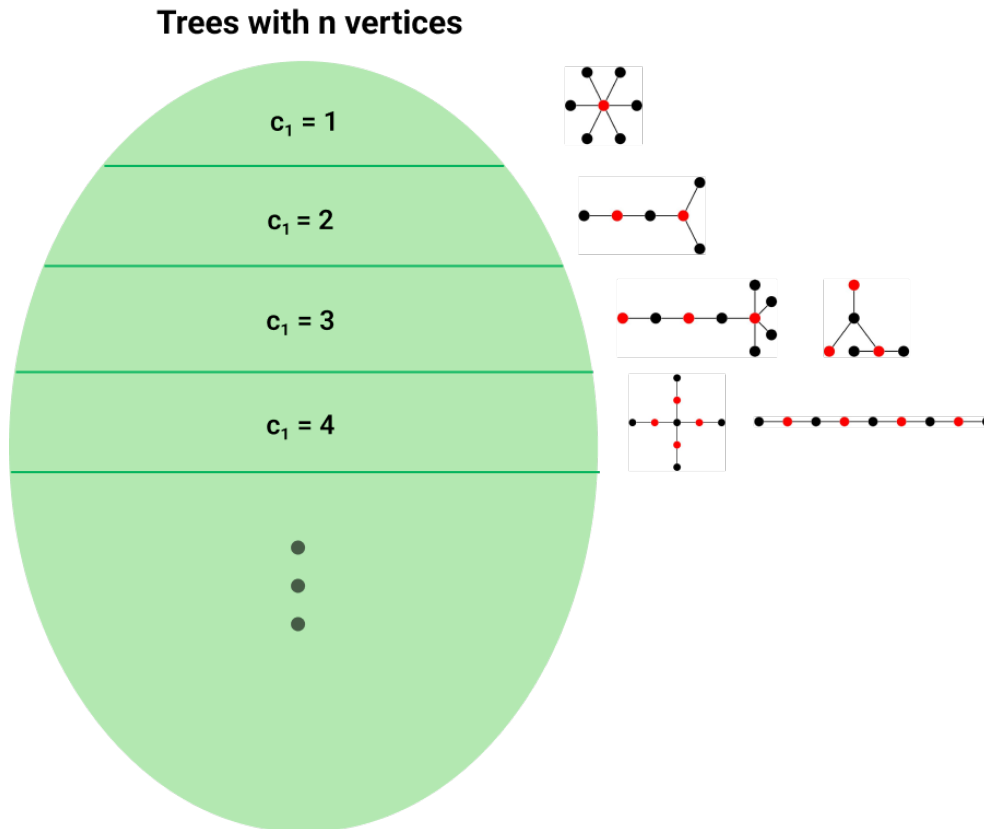
Introduction

Graph theory has various applications, ranging from theoretical mathematics to applied mathematics and computer science. One of the most popular problems in graph theory is the four-color theorem. The theorem states that any map can be colored with four colors so that no two regions sharing borders have the same color. Coloring graphs has been of particular interest to mathematicians due to its wide-ranging applications, such as scheduling problems and solving Sudoku puzzles, to name a few. In 1995, Stanley introduced the symmetric function generalization to the chromatic polynomial in [St], and chromatic symmetric functions have been widely studied since. In this thesis, I explore ways to classify graphs using their chromatic symmetric functions.

The main objective of my thesis is to restrict the number of colors used and examine whether trees can be distinguished by their chromatic symmetric functions with a limited number of colors. A tree that has a chromatic symmetric function of $24m_{1111} + 6m_{211} + 2m_{22}$ has a chromatic symmetric function of $6m_{211}$ when the number of colors is restricted to 3. While explanations of the notations in the m basis will be given in later sections, this example describes the intuition that there

are fewer ways to color a tree when there is a restriction on the number of colors used, making it less likely for trees to be distinguished by their chromatic functions.

I start my thesis by exploring two-colorings, or bicolorings, of trees, where we use exactly two colors to properly color trees with n vertices. Particularly, I examine how many times each of the two colors is used. For bicoloring of trees with n vertices, if one color is used c_1 times, then the other color is used $c_2 = n - c_1$ times. I denote by c_1 the color that was used fewer times, which will be consistent throughout this thesis. Trees with n vertices can be partitioned by the value of c_1 , as shown in the following diagram. From those partitions, I derive formulas that count the size of each block



associated with differing values of c_1 . I show that there is some combinatorial object that can be bijected to the trees satisfying the condition, and prove there exist some minimal unit(s) for each block that generate every tree with n vertices that can be

bicolored with one color used exactly c_1 times.

However, naturally, trees with n vertices are not distinguished by the 2—chromatic symmetric function. Thus, the fourth chapter in this thesis investigates how many colors need to be used to distinguish trees by their chromatic symmetric functions (distinguishing number). I outline the potential next steps in the fifth and last chapter.

Chapter 2

Preliminaries and Definitions

2.1 Combinatorial Objects

In this section, I define several combinatorial objects that I will use commonly throughout this thesis. Most terms can be found in introductory combinatorics textbooks such as [Br]. I also borrowed some definitions from [Eg]. Readers who are familiar with introductory combinatorics may skip to **Definition 3**.

Definition 1. A **partition** of integer n into m parts is a sequence of positive integers $(\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 + \dots + \lambda_m = n$ and $\lambda_1 \geq \dots \geq \lambda_m > 1$.

Definition 2. A **composition** of an integer n into m parts is a sequence of positive integers $(\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 + \dots + \lambda_m = n$. A **weak composition** may have $\lambda_j = 0$ for some j .

The main difference between a partition and a composition of an integer n is that a composition counts reordered sequences with the same elements as different objects, while a partition counts them as the same object.

Example 1. *Partitions of 6 into three parts are*

$$(1, 1, 4), (1, 2, 3), (2, 2, 2),$$

whereas compositions of 6 into three parts are

$$(1, 1, 4), (1, 4, 1), (4, 1, 1), (1, 2, 3), (1, 3, 2),$$

$$(2, 1, 3), (2, 3, 1), (3, 2, 1), (3, 1, 2), (2, 2, 2)$$

In general, it is easier to count compositions of n than count partitions of n as there are known formulas for the number of compositions of n into k parts. Now, I define a concept that derives from the definition of partitions and compositions that may not be defined in introductory combinatorics courses.

Definition 3. *A non-symmetric partition of n into k parts is a weak composition of n into k parts up to reversal.*

Let us break down this definition. Since I have a composition, I count reordered sequences with the same elements as different objects. Up to reversal means that I consider two objects the same if they are reverses of one another. For example, I consider $(1, 0, 0)$ equivalent to $(0, 0, 1)$, but $(0, 1, 0)$ is distinguished from $(1, 0, 0)$ or $(0, 0, 1)$. For a more precise demonstration of this definition, see **Example 2**.

Example 2. *For $n = 4$, partitions into at most three parts are :*

$$(4, 0, 0), (4, 1, 0), (2, 2, 0), (2, 1, 1)$$

Weak compositions of 4 into three parts are:

$$\begin{aligned} &(0, 0, 4), (0, 1, 3), (0, 2, 2), (0, 3, 1), (0, 4, 0), \\ &(1, 0, 3), (1, 1, 2), (1, 2, 1), (1, 3, 0), (4, 0, 0), \\ &(2, 0, 2), (2, 1, 1), (2, 2, 0), (3, 0, 1), (3, 1, 0) \end{aligned}$$

Lastly, non-symmetric partitions of 4 into three parts are:

$$\begin{aligned} &(4, 0, 0), (0, 4, 0), (3, 1, 0), (0, 3, 1), (1, 0, 3), \\ &(2, 2, 0), (2, 0, 2), (2, 1, 1), (1, 2, 1) \end{aligned}$$

This definition will be used in **Chapter 3**. In general, in order to use this definition, we want to put symmetric parts in opposite ends of the partition. We will discuss this definition further in the next section.

2.2 Basic graph theory

This section gives readers an overview of basic terminology in graph theory that may arise in this thesis. As the concepts introduced can be found in introductory graph theory textbooks ([Ha], [Bo], or [De]), readers who are familiar with graph theory may skip to **section 2.3**.

Definition 4. A **graph** G is a tuple $(V(G), E(G))$ where $V(G)$ represents a non-empty set of **vertices** and $E(G)$ represents a set of **edges**. Two vertices $u, v \in V(G)$ are **adjacent** if $(u, v) \in E(G)$.

I only study simple graphs in this thesis. A **simple graph** is a graph where 1) no two edges share the same start and endpoint and 2) there is no loop.

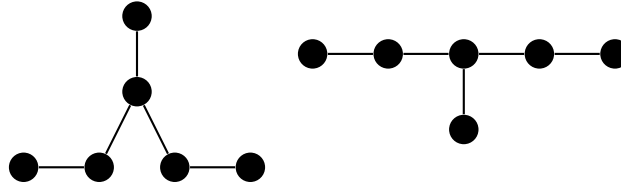


Figure 2.1: Isomorphic graphs

Definition 5. The **degree** of vertex v , $d(v)$, is the number of edges incident to v .

Definition 6. The **leaf** is a vertex with $d(v) = 1$.

Definition 7. Two graphs G and H are **isomorphic** if there is a bijection $\phi : V(G) \rightarrow V(H)$ such that $(v, w) \in E(G)$ if and only if $(\phi(v), \phi(w)) \in E(H)$ for any pair of vertices $v, w \in V(G)$.

Isomorphism, in lay terms, means the "sameness" of two or more graphs. In other words, two graphs are isomorphic when there is a one-to-one correspondence between the vertices and the edges of the graph. Even if two graphs look different in the way they are visually presented like in **Figure 2.1**, they are actually the same graph if they have a bijection ϕ as described in **Definition 6**.

Showing the isomorphism or non-isomorphism of two or more trees is important in this thesis. Two graphs cannot be isomorphic if any of those conditions are satisfied: [El]

- One graph is bipartite (the vertices can be partitioned into two) and the other is not.
- They have distinct complements.
- They have different order or size.
- One graph has a vertex of degree k and the other doesn't.

- The chromatic number is not the same.
- One contains a cycle and the other does not.
- They have different girth (the length of the graph's shortest cycle).

Definition 8. A **path** is a sequence of edges that join vertices in a graph. Moreover, we may call a graph a **path** if the graph only has one path.



Figure 2.2: Path of length 4

Definition 9. The **diameter** of a graph is the length of the longest path that can be formed between any two vertices of the graph.

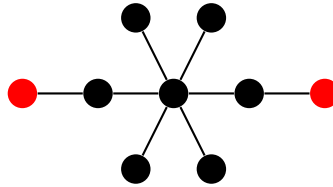


Figure 2.3: Graph with diameter 4

The diameter of this graph is 4 since the longest path that can be formed in the graph is between the leftmost vertex and the rightmost vertex marked in red. There are four edges that connect those vertices, making the diameter 4.

Definition 10. A graph G is **connected** if there is a path between any two vertices in the graph.

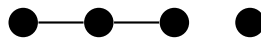


Figure 2.4: Not connected

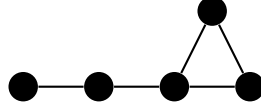


Figure 2.5: Cyclic graph.

Definition 11. A **cycle** in graph is a path that starts from one vertex and ends in the same vertex. A graph is **acyclic** if the graph does not have any cycles.

With the definition of **connected** and **cycle**, we now arrive at the definition of the **tree**.

Definition 12. A **tree** is a connected, acyclic graph.

Here, I state some proven facts (propositions) about trees. Readers may refer to introductory graph theory textbooks for the proofs of these facts.

Proposition 1. In a tree, any two vertices are connected by one path.

Proposition 2. A tree on n vertices has $n - 1$ edges.

Proposition 3. Every nontrivial tree has at least two leaves. A trivial tree (tree with one vertex) has one leaf.

The following definitions involving **proper coloring** lead to chromatic symmetric functions, which are the main subject of this thesis.

Definition 13. A **proper coloring** of graph G is a coloring of the vertices of G where no two adjacent vertices share the same color. More precisely, it is a function $\kappa : V(G) \rightarrow [k]$ where, for $u, v \in V(G)$, $\kappa(u) \neq \kappa(v)$ if $(u, v) \in E(G)$.

Here, k denotes the number of colors used for coloring.

Definition 14. A **chromatic number** of graph G is the number of proper colorings of G .

2.3 Symmetric Functions and Coloring of Graphs

We define chromatic symmetric functions consistent with Stanley's generalization of **chromatic symmetric functions** in [St]. The definitions in this section were borrowed from [St] and [Eg]. We will begin this section by defining **symmetric functions** in general terms.

Definition 15. *A **symmetric function** is a function that is not changed by the permutation of variables. That is, for commuting invariants x_1, \dots, x_n ,*

$$f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

A **symmetric polynomial** is a symmetric function that is represented in a polynomial form. For instance, $x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^2$ is a symmetric polynomial since commuting the variables does not change the polynomial. The **degree** of each monomial is $3 + 1 + 1 = 5$, which is the sum of the exponents of indeterminants. This polynomial is a **homogeneous polynomial** since it is composed of monomials with the same degree. Now, we can define **chromatic polynomial**.

Definition 16. *A **chromatic polynomial** is a function $\chi_G(k) : \mathbb{N} \rightarrow \mathbb{N}$ that represents the number of proper colorings of G using k colors. It is also the sum of all proper colorings of G with k colors, in which each term is 1.*

Notice that $\chi_G(k) = 0$ if a graph G contains a loop. If there is an edge that connects a vertex to itself, $(u, u) \in E(G)$ but $\kappa(u) \neq \kappa(u)$, which means there isn't any proper coloring of G . Moreover, if there are multiple edges sharing the same start and end vertices, removing all but one of those edges does not change $\chi_G(k)$. Thus, it is customary to only consider simple graphs when discussing chromatic polynomials.

The following proposition in [Eg] gives a formula for the chromatic polynomial of trees.

Theorem 1. *If T is a tree with n vertices, then $\chi_T(k) = k(k-1)^{n-1}$ for all $k \geq 1$.*

Proof. We can prove this by induction on n . $n = 1$ is trivial since there are k ways to color a single vertex with k colors. Now suppose $n \geq 1$. Assume the equation holds for trees with $n - 1$ vertices. By **Proposition 3**, T with n vertices has two leaves. If we remove one of these vertices, denoted v' , then we have a tree T' with $n - 1$ vertices. By our assumption, we can color T' with k colors in $k(k-1)^{n-2}$ ways. The choice of coloring v' is $k - 1$ colors. v' has one neighbor, and we can color v' with any color but the color of the neighbor's. Thus, we get $\chi_T(n) = k(k-1)^{n-1}$. The theorem follows by the induction. \square

The graph in **Figure 2.3** is a tree and has chromatic polynomial of $k(k-1)^8$. Such is true for any tree with 9 vertices. We now discuss **chromatic symmetric functions**, which is a generalization of the chromatic polynomial of a graph by [St]. Given the definition of chromatic polynomials as the sum over all proper colorings of the graph G with k colors, we may define chromatic symmetric functions in a similar manner using **weights** of a proper coloring of G .

Definition 17. [Eg] *If c is a proper coloring of G , the **weight** of c , denoted $wt_G(c)$, is defined*

$$wt_G(c) = \prod_{v \in V} x_{c(v)}.$$

Definition 18. [Eg] *The **chromatic symmetric function** of G is defined*

$$X_G = \sum_c wt_G(c) = \sum_c \prod_{v \in V} x_{c(v)}$$

where c is a proper coloring of G with any number of colors.

Intuitively, X_G is symmetric for any graph. For some coloring c of graph G , you can permute the colors used in c . [Eg] Our choice of color is arbitrary in mathematical terms. The below proposition is the generalization of chromatic polynomial, which gives the relationship between chromatic polynomials using k colors and chromatic symmetric functions.

Proposition 4. [St] *The chromatic polynomial $\chi_G(k)$ can be defined as*

$$\chi_G(k) = X_G(1, 1, \dots, 1, 0, 0, \dots)$$

where there are exactly k 1's.

We may write chromatic symmetric functions in terms of monomials by using the definition of **stable partitions**.

Definition 19. [Eg] *A stable partition of G is the ordered set partition V_1, V_2, V_3, \dots where two adjacent vertices must belong to different blocks of the partition. (V_j may be empty.)*

For each proper coloring c of G , there is a stable partition where a partition V_j is the set of vertices of color j [Eg]. This relationship is bijective, meaning each proper coloring can produce a stable partition and vice versa. Thus, we have the following proposition.

Proposition 5. [Eg] *For any graph G with d vertices and any partition $\lambda \vdash d$, let $z_\lambda(G)$ be the number of stable partitions of G with $|V_j| = \lambda_j$ for all j . Then we have*

$$X_G = \sum_{\lambda \vdash d} z_\lambda(G) m_\lambda$$

Using this definition, we can derive a recursive formula for X_{P_n} for P_n (path of length n).

Proposition 6. *For all $n \geq 0$,*

$$X_{P_n} = e_n + \sum_{k=2}^n (k-1)e_k X_{P_{n-k}}$$

The proofs for **Proposition 5** and **Proposition 6** can be found in [Eg].

Chapter 3

Classifying Trees by their Bicoloring

In this chapter, we first show that trees with n vertices can be partitioned by their bicoloring, as shown in the green "egg-like" figure in the introduction. We examine properties of the chromatic symmetric functions of trees that can be bicolored. We begin by defining terms to be used in this chapter.

Definition 20. *A **two-coloring** of a tree refers to a proper coloring of the tree that uses exactly two colors.*

Definition 21. *A **2-chromatic symmetric function**, or **2-CSF**, of a tree refers to the chromatic symmetric function that uses exactly two colors.*

While we are mainly interested in 2-chromatic symmetric functions in Chapter 3, we can generally define **k-chromatic symmetric function** as a chromatic symmetric function that uses exactly k colors. In Chapter 4, we will look at 3-chromatic symmetric functions.

We denote by x_1 and x_2 the variables representing the colors used for the proper coloring of the tree. Then the 2-chromatic symmetric function of the tree can be represented in the form $X_T(2) = x_1^{c_1}x_2^{c_2} + x_1^{c_2}x_2^{c_1}$ where c_1, c_2 denote the number of

times the color was used and $c_1 \leq c_2$.

For a tree with $n \geq 2$ vertices, $c_1 + c_2 = n$ because if one of the colors was used for c_1 vertices, then the other color must be used for $n - c_1$ vertices. Note that the only graphs with n vertices that can be bicolored are bipartite graphs.

Theorem 2. *Trees with n vertices can be partitioned by their 2-chromatic symmetric function, as shown in this diagram.*

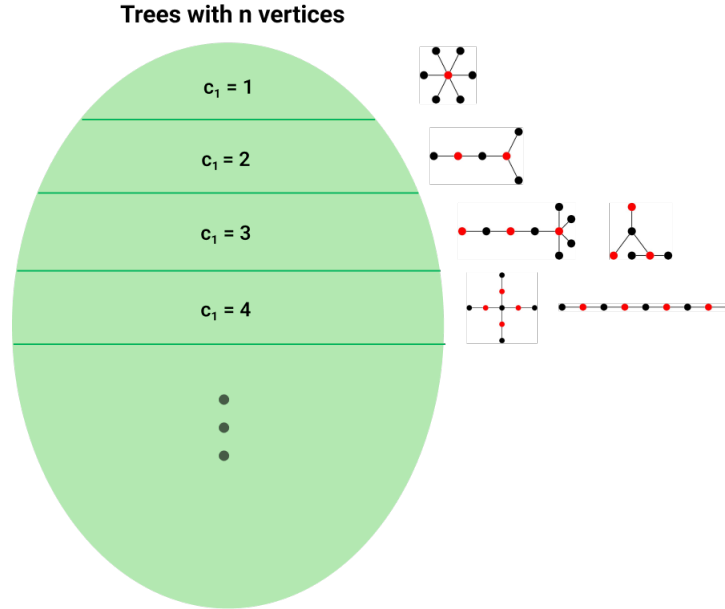


Figure 3.1: Trees with n vertices partitioned by 2-chromatic symmetric function.

Table 3.1 shows that trees with n vertices can be partitioned by the value of c_1 , computed for trees with $n \leq 21$ vertices. The columns of the table refer to trees with n vertices, and the rows of the table refer to the value of c_1 . For example, the entry for $n = 10$, $c_1 = 3$ is 19, which means that there are exactly 19 trees with 10 vertices with 2-chromatic symmetric function of $x_1^3 x_2^7 + x_1^7 x_2^3$. The last row of the table is the total number of trees with n vertices.

Since we defined $c_1 \leq c_2$, the values for $c_1 = l$ is computed only for trees with $2l$ vertices or more. In the next few sections, we will examine closely each case

Table 3.1: Partitioning Trees with n vertices.

$c_1 \backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
3					3	7	10	14	19	24	30	37	44	52	61	70	80	91	102	114
4							9	28	45	73	105	152	204	274	351	450	556	688	829	999
5									37	132	242	412	660	1008	1479	2100	2900	3911	5166	6704
6											168	693	1349	2472	4219	6890	10733	16201	23641	33689
7													895	3927	8105	15553	28089	48221	79283	125622
8															5097	23557	50517	101252	191045	343102
9																	30983	147581	326893	677790
10																			196096	956474
Total	1	1	2	3	6	11	23	47	106	235	551	1301	3159	7741	19320	48629	123867	317955	823065	2144505

concerning the value of c_1 . Readers may understand the next sections as examining the trends in each row, where **Section 3.1** refers to the row $c_1 = 1$, **Section 3.2** to row $c_1 = 2$, **Section 3.3** to row $c_1 = 3$, and **Section 3.4** to row $c_1 = 4$. **Section 3.5** will then generalize these cases. In examining each case, this chapter will also show that we can partition each block corresponding to the c_1 value even further, using **minimal units**. As the term **minimal unit** will come up frequently in this chapter, we give a formal definition in this section.

Definition 22. *A minimal unit for a block in a partition of trees is a tree that can generate every tree that belongs in that block. A tree is not a minimal unit if it can be produced from another tree belonging to that block.*

In general, there are three traits for a minimal unit. First, the minimal unit must generate every tree that belongs in the block. Second, if there are more than one minimal unit for a certain block, the minimal units should generate disjoint set of trees. Third, a minimal unit cannot be generated from another tree that belongs in that block. The idea is generally depicted in the diagram **Figure 3.2**.

When "producing" or "generating" trees from a minimal unit, we want to keep the number of red vertices. Thus, to produce a tree with n vertices from a minimal unit consisting of k vertices, we must add $n - k$ vertices, but we can only add black vertices adjacent to the red vertices since we cannot add more red vertices.

For example, **Figure 3.3** is not a minimal unit for $c_1 = 2$ case since there exists another tree that can produce this tree without adding any vertex corresponding to c_1 , as shown in **Figure 3.4**. We will show with more examples how the minimal units may apply for each case. We will then generalize more properties of minimal units in **Section 3.5**.

For the next few sections, please review the convention that $c_1 + c_2 = n$, $c_1 \leq c_2$,

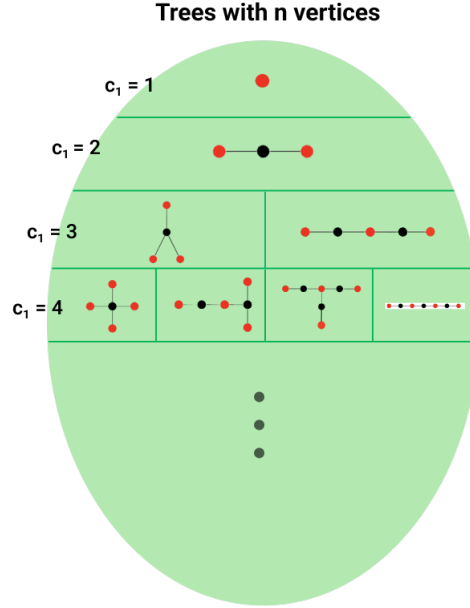


Figure 3.2: Partitioning by minimal units

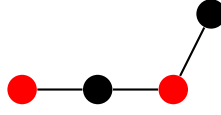


Figure 3.3: A non-example of minimal unit



Figure 3.4: A more minimal tree than **Figure 3.1**

where $x_1^{c_1}x_2^{c_2} + x_1^{c_2}x_2^{c_1}$. We are basing each case on c_1 . Moreover, let us mark the color with c_1 occurrences as red, without loss of generality since the colors can be swapped.

3.1 One color used exactly once

Let $c_1 = 1$, meaning the color with fewer occurrences is used exactly once. Trees belonging to this block have the 2-chromatic symmetric function

$$X_T(2) = x_1^1x_2^{n-1} + x_1^{n-1}x_2^1.$$

Figure 3.5 is a tree with the 2-chromatic symmetric function $x_1^1 x_2^3 + x_1^3 x_2^1$.

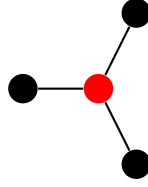


Figure 3.5: Tree with $c_1 = 1$

Proposition 7. *Any tree with n vertices that have the 2- chromatic symmetric function $x_1^1 x_2^{n-1} + x_1^{n-1} x_2^1$ is a star.*

Proof. Suppose not. Then there exists a tree T that is not the star and has the chromatic symmetric function $x_1^1 x_2^{n-1} + x_1^{n-1} x_2^1$. Since a star with n vertices has $n - 1$ leaves, this means that there exists a tree that has two or more non-leaf vertices that satisfies the condition. Denote by v_1 and v_2 two of the internal vertices of T , meaning v_1 and v_2 have degree 2 or higher. If $(v_1, v_2) \in E(T)$, then v_1 and v_2 must have different colors, namely x_1 and x_2 . However, both v_1 and v_2 must each be connected to at least one other vertex; whichever vertex connected to v_1 will have color x_2 , and whichever vertex connected to v_2 will have color x_1 . Thus, both x_1 and x_2 are used at least twice. Otherwise, if $(v_1, v_2) \notin E(T)$, then v_1 and v_2 are connected to at least two other vertices each. If v_1 and v_2 are colored the same with color x_1 , then x_1 is used at least twice. In addition, the combine of at least 4 adjacent vertices of v_1 and v_2 will be colored with the other color x_2 . If v_1 and v_2 are each colored x_1 and x_2 , then there will also be at least two adjacent vertices of v_1 colored x_2 and at least two of v_2 colored x_1 . Thus, both x_1 and x_2 are used at least three times. By contradiction, there can only be one non-leaf vertex. \square

3.1.1 Minimal unit for $c_1 = 1$

As we showed that the only trees with n vertices that can be bicolored with one color used exactly once (and the other color $n - 1$ times) are stars, the **minimal unit** for the $c_1 = 1$ case is a trivial tree with one vertex as in **Figure 3.6**.



Figure 3.6: Minimal unit for $c_1 = 1$

Any tree with 2-CSF $x_1^1 x_2^{n-1} + x_1^{n-1} x_2^1$ can be produced by adding $n - 1$ black vertices to this unit, and is a star. Stars are an important unit in the study of chromatic symmetric functions. We will see in later sections that there are more complex minimal units for larger c_1 values. Now, we consider trees that are not stars.

3.2 One color used exactly twice

Consider $c_1 = 2$, meaning the color with fewer occurrences is used exactly twice. This means that the 2-CSF for trees in this block is

$$X_T(2) = x_1^2 x_2^{n-2} + x_1^{n-2} x_2^2.$$

Figure 3.7 is an example of a tree with 6 vertices where $c_1 = 2$.

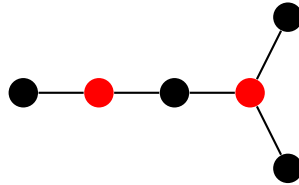


Figure 3.7: Tree with $c_1 = 2$

While we did not state explicitly in the case for $c_1 = 1$, the maximum diameter of

trees that satisfy the condition $c_1 = 1$ is 2 since all such trees are stars. **Proposition 8** states a similar condition for $c_1 = 2$.

Proposition 8. *All trees with n vertices that can be bicolored with one of the colors used exactly two times (and the other color $n - 2$ times) have diameter at most 4.*

Proof. Suppose that there is a tree that satisfies the condition $c_1 = 2$ with a diameter greater or equal to 5. A tree with diameter greater or equal to 5 contains a path with at least 6 vertices. Since x_1 and x_2 must alternate, both x_1 and x_2 must be used exactly three times in the path with 6 vertices, contradicting our condition. \square

3.2.1 Minimal unit for $c_1 = 2$

To derive the minimal unit for $c_1 = 2$ case, let us give more examples of trees with this 2-CSF.

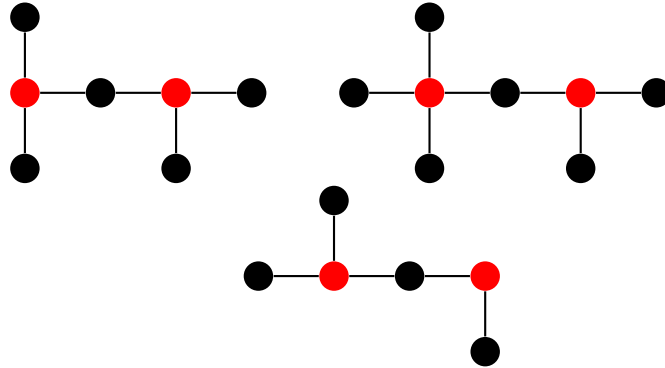


Figure 3.8: Examples for $c_1 = 2$

Notice that for all trees satisfying $c_1 = 2$, we have a spine of length 3. This is because the two vertices colored in red must be one vertex apart from each other. The red end vertices of the spines are connected to black leaf vertices. Therefore, for

$c_1 = 2$, the **minimal units** are the spines with 3 vertices. Just like the $c_1 = 1$ case, there is only one minimal unit for $c_1 = 2$ case.



Figure 3.9: Minimal Unit for $c_1 = 2$

The intuition behind **Theorem 3** is that there are $n - 3$ leaf vertices connected to the two red vertices.

Theorem 3. Denote by $g(n)$ the number of proper colorings of trees with n vertices with 2-CSF

$$X_T(2) = x_1^2 x_2^{n-2} + x_1^{n-2} x_2^2.$$

Then $g(n)$ is the number of partitions of $n - 3$ into at most two parts.

Proof. Let \mathbf{T} be the set of trees that satisfy this condition and \mathbf{P} the set of a partition of $n - 3$ into two parts. Then $f : \mathbf{T} \rightarrow \mathbf{P}$ maps $t \in \mathbf{T}$ to partitions of $n - 3$ where the end vertices of the spine, v_1 and v_2 , are each connected to a_1 and a_2 leaves, which corresponds to the partition $(a_1, a_2) \in \mathbf{P}$. Notice that $a_1 + a_2 = n - 3$ since the spine has length 3. To show $|\mathbf{T}| = |\mathbf{P}|$, we need to show that f is a bijective function.

Suppose toward a contradiction that f is not injective. Then for some partition (a_1, a_2) of $n - 3$ into two parts, there exist two non-isomorphic trees. Then $f(t_1) = f(t_2) = (a_1, a_2)$, where $t_1 \neq t_2$. On the three-vertex spine of t_1 , the number of leaves connected to one of the end vertices is a_1 and the number of leaves connected to the other end vertex is a_2 . However, the same is true for t_2 . Since the tree is unlabeled, t_1 and t_2 are isomorphic. Thus we have a contradiction and f is injective. Surjectivity also follows intuitively by construction of f . We need to show that for all $(a_1, a_2) \in \mathbf{P}$, every tree that can be constructed by adding a_1 and a_2 leaves to the end vertices of the spine is a tree that belongs to \mathbf{T} . This follows since the end vertices of the spine

can be colored with one color that is used exactly $c_1 = 2$ times and the rest with the other color that is used exactly $n - 2$ times since $a_1 + a_2 + 1 = n - 2$, where the 1 corresponds to the middle vertex of the spine. \square

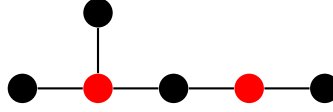


Figure 3.10: $(a_1, a_2) = (2, 1)$

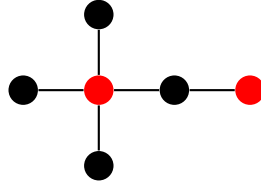


Figure 3.11: $(a_1, a_2) = (3, 0)$

Theorem 3 is backed up by the sequence A004526 on Online Encyclopedia of Integer Sequences [OE1], which corresponds to the row with $c_1 = 2$ in **Table 3.1**. Moreover, using the bijection, we can count the number of bicolourings of trees with n vertices where $c_1 = 2$ and $c_2 = n - 2$.

Corollary 1. *The number of bicolourings of trees with $n \geq 4$ vertices where $c_1 = 2$ is*

$$g(n) = \left\lfloor \frac{n-1}{2} \right\rfloor$$

Proof. Algebraically,

$$g(n) = \left\lfloor \frac{n-1}{2} \right\rfloor = 1 + \left\lfloor \frac{n-3}{2} \right\rfloor \quad (3.1)$$

The number of partitions of $n - 3$ into one part is 1, corresponding to the 1 in (3.1).

Then, we want to count the partitions of $n - 3$ into two parts, and we can count it

by the number of possible smallest part in each partition. In other words, in (a_1, a_2) where $a_2 \leq a_1$, how many values are possible for a_2 . We can derive that for each n , the partitions can range from $(n-4, 1)$ to $(\lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor)$ or $(\lfloor \frac{n-3}{2} \rfloor + 1, \lfloor \frac{n-3}{2} \rfloor)$. For odd n , the maximum happens when the two parts are equal $(\lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor)$, and for even n , it happens when the partition is $(\lfloor \frac{n-3}{2} \rfloor + 1, \lfloor \frac{n-3}{2} \rfloor)$. Thus, there are $\lfloor \frac{n-3}{2} \rfloor$ partitions of $n-3$ into exactly 2 parts, making our complete formula

$$g(n) = 1 + \left\lfloor \frac{n-3}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

□

3.3 One color used exactly three times

Let's consider the case where $c_1 = 3$, meaning the color with fewer occurrences was used exactly three times. Trees in this case have the 2-CSF

$$X_T(2) = x_1^3 x_2^{n-3} + x_1^{n-3} x_2^3.$$

Figure 3.12 is an example in which one color was used exactly three times.

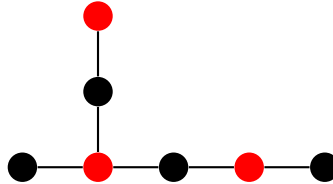


Figure 3.12: Tree where $c_1 = 3$

Proposition 9 is analogous to **proposition 8** of **Section 3.2**. We will skip the proof since the proof is exactly the same, except that you replace the path of length

5 with length 7, meaning that the path contains 8 vertices, and show that each color must be used 4 or more times.

Proposition 9. *All trees that can be bicolored with one of the colors used exactly three times have diameter at most 6.*

3.3.1 Minimal Units for $c_1 = 3$

For the bicoloring of trees where one of the colors is used exactly three times, the **minimal units** are a star with four vertices and a path with five vertices as described in **Figure 3.13**. For convenience, we refer to the first minimal unit as "the path" and the second minimal unit as "the star."

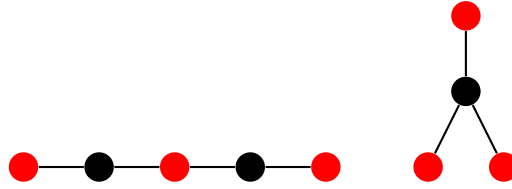


Figure 3.13: Path unit (left) and Star unit (right)

• Path unit for $c_1 = 3$

Let's first consider the path unit. Among trees with n vertices that have the 2-CSF

$$X_T(2) = x_1^3 x_2^{n-3} + x_1^{n-3} x_2^3,$$

some trees can be produced from the path unit by adding black vertices to the three red vertices.

Theorem 4. *Let $f(n)$ be the number of trees with the above 2-CSF that can be produced from the path unit. Then $f(n)$ is the number of nonsymmetric partitions of $n - 5$ into three parts.*

$f(n)$ is the sequence A002620 on Online Encyclopedia of Integer Sequences, and the term non-symmetric sequence was mentioned by Jon Perry in [OE3].

Similar to the proof of **Theorem 3**, we can construct a function f from the set of trees satisfying this condition to the set of nonsymmetric partitions of $n - 5$. Showing the bijectivity of this function is analogous to showing the bijectivity of f in **Theorem 3**.

Example 3. When $n = 6$, you have two non-isomorphic graphs that are produced from the path.

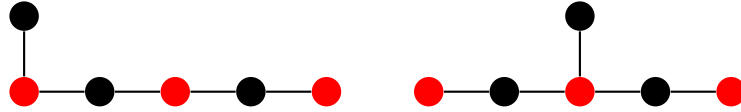


Figure 3.14: Trees 6 vertices produced from path

We denote those trees $(1, 0, 0)$ and $(0, 1, 0)$, respectively. Note that we cannot add another red vertex to the rightmost red vertex as the tree will be isomorphic to the first tree. In other words, $(0, 0, 1)$ is isomorphic to $(1, 0, 0)$ since the partitions are the same if reversed.

Example 4. When $n = 9$, we have nine trees that can be built from the path unit.

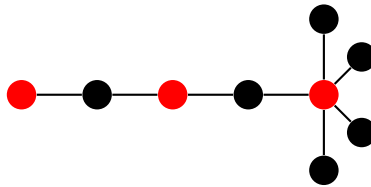


Figure 3.15: $(0, 0, 4)$

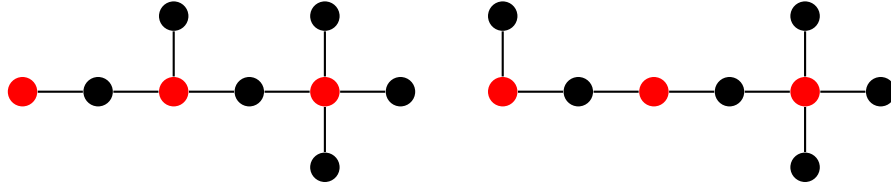


Figure 3.16: $(0, 1, 3)$ and $(1, 0, 3)$

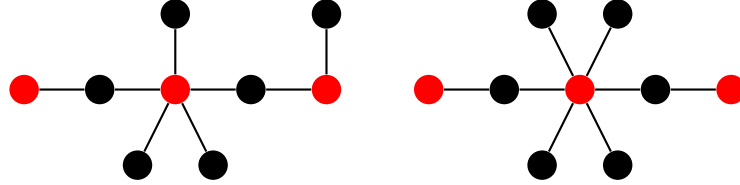


Figure 3.17: $(0, 3, 1)$ and $(0, 4, 0)$

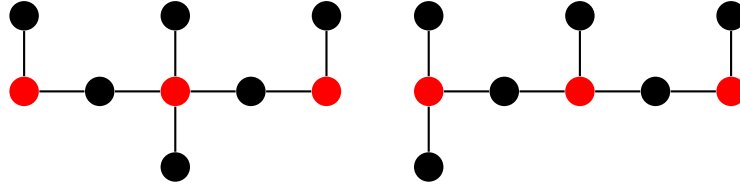


Figure 3.18: $(1, 2, 1)$ and $(2, 1, 1)$

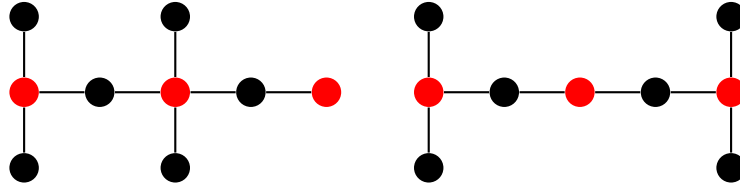


Figure 3.19: $(2, 2, 0)$ and $(2, 0, 2)$

Note the non-symmetric partitions corresponding to each tree. As described from the examples, each bicoloring of tree using one color exactly three times generated from the path unit correspond to the nonsymmetric partitions of $n - 5$ into at most three parts, with zeros used for padding. **Theorem 4** leads to a corollary that count the exact number of trees that belong to this case. The formula is derived from the sequence A002620 on Online Encyclopedia of Integer Sequences [OE3].

Corollary 2. *The number of bicoloring of trees with n vertices that can be produced*

from the path unit and uses one color exactly three times is the quarter square of $n-3$:

$$\left\lfloor \frac{(n-3)^2}{4} \right\rfloor.$$

• **Star unit for $c_1 = 3$**

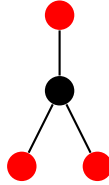


Figure 3.20: Star unit

Let's direct our attention to the **star unit**. Some trees with n vertices that have the 2-CSF corresponding to the case $c_1 = 3$ can be generated from the star unit. Since the star is symmetric about the center, adding a vertex to any of the red vertices produces an isomorphic graph.

Theorem 5. *Let $h(n)$ denote the number of trees with n vertices that have the 2-CSF corresponding to the case $c_1 = 3$ and can be generated from the star unit. Then $h(n)$ is the number of partitions of $n - 4$ into at most three parts.*

Proof. We show that there is a bijection relationship between trees with this condition and partitions of $n - 4$ into at most three parts. The proof is analogous to the proof of **Theorem 3** and **Theorem 4**. □

Example 5. *When $n = 6$, there are two non-isomorphic trees that can be produced from the star unit. Notice $(1, 1, 0)$ is isomorphic to $(0, 1, 1)$ or $(1, 0, 1)$.*

$h(n)$ corresponds to the sequence A001399 in [OE4]. In the sequence A001399, $a(n - 4) = h(n)$. Corollaries to **Theorem 5** count the exact number of trees $h(n)$.

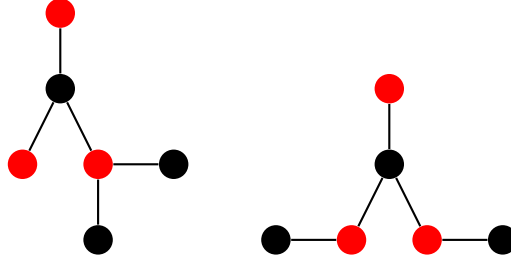


Figure 3.21: $(2, 0, 0)$ (left) and $(1, 1, 0)$ (right)

Corollary 3. $h(n)$ is also equivalent to the number of partitions of $n - 1$ into exactly three parts.

Corollary 3 follows from **Theorem 5** when instead of considering the number of vertices attached to each red vertex, we include the red vertex in the size of each block, making the size of each block non-zero. We are only excluding the center vertex in the partition, creating partitions of $n - 1$ into exactly three parts.

In the next corollary, we consider a special case of trees: tripods. **Tripods** are trees with exactly three leaves, and there is exactly one vertex with degree 3 called the **hub**. In **Figure 3.22**, the three leaves are marked in red, and the hub is marked in yellow.

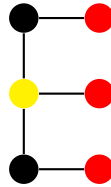


Figure 3.22: Tripod With 6 Vertices

Corollary 4. There is a bijection between trees with n vertices that can be produced from stars where $c_1 = 3$ and tripods with n vertices [OE4].

Proof. Excluding the hub, let the vertices be partitioned based on their relative location to the hub. That is, since the hub has degree 3, the vertices are partitioned by

which of the three edges they are connected to. From that, we can derive a bijection with partitions of $n - 1$ into exactly three parts, which are bijective to the trees with n vertices that can be produced from stars where $c_1 = 3$ by **Corollary 3**. \square

In **Table 3.2**, the red portion of the tree represents the star unit, showing a correspondence between two classes of trees.

Corollary 5. *The partition of trees with n vertices and $c_1 = 3$ produced from the star unit is*

$$h(n) = \text{round} \left(\frac{(n-1)^2}{12} \right).$$

Proof. We have shown the bijection relationship between partitions of $n - 4$ into at most three parts with zeros used as padding and the trees with $a = 3$ that uses the star unit as the minimal unit. Hardy G.H proved in [Hd] that partitions of k into at most three parts corresponds to the formula $\text{round} \left(\frac{(k+3)^2}{12} \right)$. Substituting $k = n - 4$, we get the above formula. \square

Corollary 6. *The number of trees with $c_1 = 3$ that uses the star unit as the minimal unit is given by the generating function*

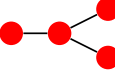

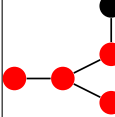
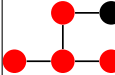
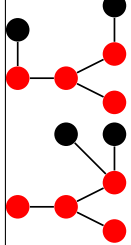
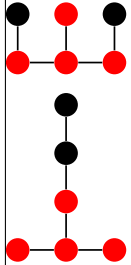
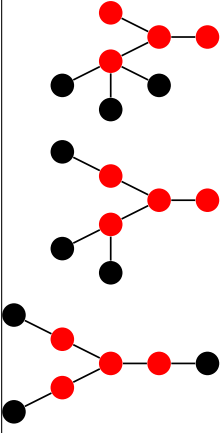
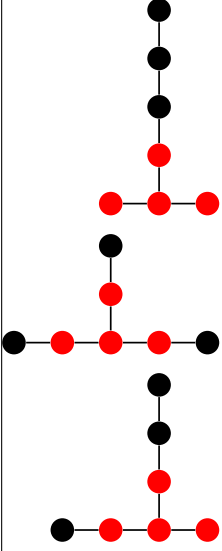
$$\frac{x^3}{(1-x)(1-x^2)(1-x^3)} = x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + \dots,$$

which corresponds to the number of partitions of $n - 4$ into at most three parts [OE4].

In the above generating function, the number of trees with $n = 6$ vertices correspond to the coefficient 2 of x^5 . In general, the number of trees with n vertices correspond to the coefficient of x^{n-1} . See the below examples using trees with 6 vertices.

Generally, we define minimal units to claim that those units further partition the block with $c_1 = 3$. However, does that work for every n ? For trees with 6 vertices

Table 3.2: Tripods, Star Unit, and Partitions of $n - 1$

Tree star unit	Tripods	Partition of $n-1$
		$(1,1,1)$
		$(2,1,1)$
		$(2,1,1), (3,1,1)$
		$(3, 2, 1), (2, 2, 2), (4, 1, 1)$

that have the 2-CSF

$$X_T(2) = x_1^3 x_2^{n-3} + x_1^{n-3} x_2^3,$$

a tree $(1, 1, 0)$ that can be produced from the star unit is isomorphic to the tree $(0, 1, 0)$ that can be produced from the path unit.



Figure 3.23: Isomorphic trees with 6 vertices

This is because trees with 6 vertices is equicolorable when $c_1 = 3$, meaning when one color is used exactly three times, the other color is necessarily used three times as well. However, we claim that for trees with 7 or more vertices, the minimal units star and path produce disjoint sets of trees that belong in that block.

Lemma 1. *For $n \geq 7$, bicoloring of trees where $c_1 = 3$ that can be produced from the star unit are disjoint from the trees that can be produced from the path unit.*

Proof. Suppose toward a contradiction that there is an intersection between the trees that can be produced from the star unit and the trees that can be produced from the path unit. Trees that can be produced from the path unit have diameter at least 4 and at most 6, whereas trees that can be produced from the star unit have diameter at least 2 and at most 4. Thus, in order for a tree to belong to the intersection, the only viable diameter of the tree is 4.



Figure 3.24: Path has diameter at least 4

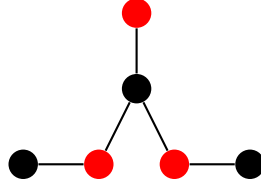


Figure 3.25: Star has diameter at most 4

Now suppose there is a tree with $n \geq 7$ vertices with diameter 4 that can be produced from both the path unit and the star unit. Notice that **Figure 3.25**, which represents the star unit with diameter 4, contains a path of length 4 consisting of vertices alternating between black and red. However, while **Figure 3.24** has two red leaves in the path, **Figure 3.25** has two black leaves in the path. For $n \geq 7$ vertices, if $c_1 = 3$, $c_2 = n - 3$ and $n - 3 \neq 3$. Thus, if the places of red and black vertices are different, then the trees cannot be isomorphic while having $c_1 = 3$. \square

We have proved that for $n \geq 7$, the star unit and the path unit produce nonisomorphic trees. We also know that the two units combined produce all trees for the $c_1 = 3$ case. The star unit sequence is the number of partitions of $n - 4$ into at most three parts. For some small n values, we get the sequence 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19, 21, ..., which is in [OE4].

Then, the path sequence is the number of nonsymmetric partitions of $n - 5$ into at most three parts. For some small n values, we get 1, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49..., as defined in [OE3].

Combining those two, we get the sequence 2, 2, 4, 7, 10, 14, 19, 24, 37, 44, 52, 61, 70.... Since the first two sequences correspond to trees with 4 and 5 vertices, we must exclude those from the sequence as c_1 cannot be 3 while having $c_1 \leq c_2$ as required. Then we have $a(n) = 4, 7, 10, 14, 19, 24, 37, 44, 52, 61, 70, \dots$. This sequence is A007980 on OEIS [OE2]. $a(0)$ corresponds to the number of trees with $n = 6$ vertices in the

$c_1 = 3$ block, and we have shown that there is a tree that is at the intersection of the star and the path unit due to being equicolorable. Thus, $a(0) = 4$ even though there are 3 trees with 6 vertices for the $c_1 = 3$ case. Starting from trees with $n \geq 7$ vertices and $a(1)$, the sequence corresponds to the total number of bicoloring of trees with n vertices where $c_1 = 3$.

Let us denote by $g(n)$ the total number of trees with n vertices that can be bicolored with $c_1 = 3$.

Theorem 6. *The number of trees with n vertices that can be bicolored with $c_1 = 3$ that uses the star unit is (shown in [OE4])*

$$\left\lfloor \frac{(n-1)^2 + 4}{12} \right\rfloor,$$

and the partition of such trees that uses the path unit as the minimal unit is

$$\left\lfloor \frac{(n-3)^2}{4} \right\rfloor.$$

Thus,

$$g(n) = \left\lfloor \frac{(n-1)^2 + 4}{12} \right\rfloor + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor = \left\lfloor \frac{n^2 - 5n + 7}{3} \right\rfloor$$

for the total number of trees with n vertices that can be bicolored with $c_1 = 3$.

This formula $g(n) = \left\lfloor \frac{n^2 - 5n + 7}{3} \right\rfloor$ is given by [OE2], which corresponds to the sequence for the total number of trees with n vertices where the 2-CSF is

$$X_T(2) = x_1^3 x_2^{n-3} + x_1^{n-3} x_2^3.$$

Algebraically, the equation makes sense by the calculation below.

$$\begin{aligned}
\left\lfloor \frac{(n-3)^2}{4} \right\rfloor + \left\lfloor \frac{(n-1)^2 + 4}{12} \right\rfloor &= \left\lfloor \frac{n^2 - 6n + 9}{4} \right\rfloor + \left\lfloor \frac{n^2 - 2n + 5}{12} \right\rfloor \\
&= \left\lfloor \frac{3n^2 - 18n + 27}{12} \right\rfloor + \left\lfloor \frac{n^2 - 2n + 5}{12} \right\rfloor \\
&= \left\lfloor \frac{n^2 - 5n + 8}{3} \right\rfloor \\
&= \left\lfloor \frac{n^2 - 5n + 7}{3} \right\rfloor
\end{aligned} \tag{3.2}$$

The last two lines of equation 3.2 are equal because

$$n^2 - 5n = n(n-5) \equiv m \in \{0, 1\} \pmod{3}.$$

We know $5 \equiv 2 \pmod{3}$ and thus if $n \equiv i \pmod{3}$, then $n-5 \equiv i+1 \pmod{3}$. If $i = 0$ or $i+1 = 0$, then $n(n-5) \equiv 0 \pmod{3}$. Otherwise, $n(n-5) \equiv 2 \pmod{3}$.

On the other hand, if $k \equiv 1 \pmod{3}$ and $j_1 \cdot j_2 = k$, then $j_1 \equiv j_2 \equiv m \pmod{3}$ where $m \in \{1, 2\}$ so that $j_1 \cdot j_2 \equiv 1 \pmod{3}$.

Now, the sequence A007980 on OEIS [OE2] that gave us the formula $g(n) = \left\lfloor \frac{n^2 - 5n + 7}{3} \right\rfloor$ also leads us to **Corollary 7** and **Corollary 8**.

Corollary 7. *For $n \geq 7$, $g(n)$ is the number of partitions of $2 \cdot (n-4)$ to at most three parts. It is also the number of partitions of $2n-5$ into exactly three parts.*

Corollary 8. *For $n \geq 7$, $g(n)$ is equal to the expansion of*

$$\frac{(1+x^2)}{(1-x)^2(1-x^3)}.$$

It is the generating function for the number of partitions of $2 \cdot (n-4)$ to at most three parts and number of partitions of $2n-5$ into exactly three parts [OE2].

From the sequence, one may also come up with the recursive formula for $g(n)$ that we leave as a conjecture.

Conjecture 1. $g(n)$ can also be defined recursively as the following.

$$g(7) = 7$$

$$g(8) = 10$$

Now, for $n \geq 9$,

1. if $n \equiv 0 \pmod{3}$ or $n \equiv 1 \pmod{3}$,

$$g(n) = 2 * g(n - 1) - g(n - 2) + 1$$

2. if $n \equiv 2 \pmod{3}$,

$$g(n) = 2 * g(n - 1) - g(n - 2)$$

The idea of the proof may be related to the minimal unit relationship.

3.4 One color used exactly four times

Lets now direct our attention to the case where $c_1 = 4$, meaning one of the colors used was used exactly four times and the other color $c_2 = n - 4$ times. Trees in this case have the 2-CSF

$$X_T(2) = x_1^4 x_2^{n-4} + x_1^{n-4} x_2^4.$$

Figure 3.26 an example where one color was used exactly four times.

The first proposition in this section is analogous to **Proposition 9** in **Section 3.3** and **Proposition 8** in **Section 3.2**. Thus, we leave an abridged proof.

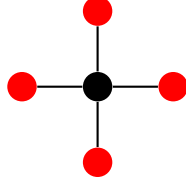


Figure 3.26: Tree with $c_1 = 4$

Proposition 10. *All trees that can be bicolored with one of the colors used exactly four times ($c_1 = 4$) and the other color $n - 4$ times have diameter at most 8.*

Proof. Suppose that there is a tree with n vertices with diameter 9 or higher that can be properly bicolored with $c_1 = 4$. In any tree with diameter 9 or more, there exists a path with 10 vertices. Since colors have to alternate, c_1 and c_2 have to be used the equal number of times in the path, which means each color is used 5 times or more. □

3.4.1 Minimal units for $c_1 = 4$

We have four minimal units for $c_1 = 4$, and we call them the star, the path, the T, and the trident as depicted in **Figure 3.27**. We show that those units are minimal units, meaning 1) every tree with 2-CSF $X_T(2) = x_1^4 x_2^{n-4} + x_1^{n-4} x_2^4$ can be generated from these units and 2) the set of trees that can be generated from each unit is disjoint from one another.

Following the convention that $c_1 \leq c_2$ and that trees with $n = 8$ vertices will have intersection between trees that can be produced from minimal units because they are equicolorable, we want to show the disjointness for such trees with $n \geq 9$ vertices.

Theorem 7 shows the disjointness.

Theorem 7. *Trees with n vertices that can be produced from star, path, T, and trident units are disjoint to each other for all $n \geq 9$.*

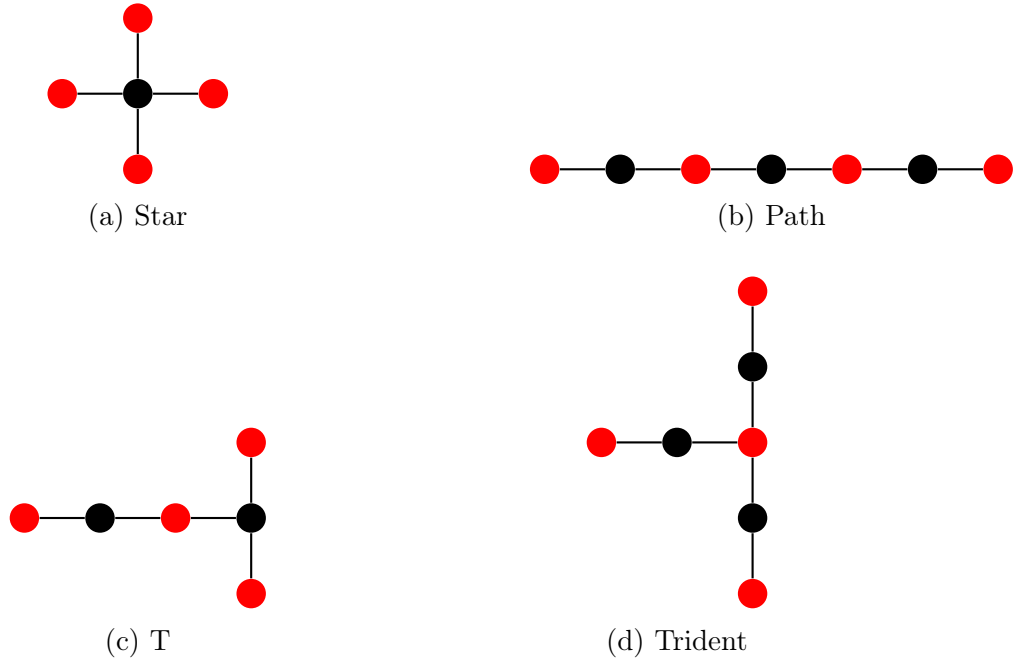


Figure 3.27: Minimal Units for $c_1 = 4$

Proof. In order to show that all of them are disjoint from one another, we must show 1) star is disjoint to path, 2) path is disjoint to T, 3) T is disjoint to trident, 4) star is disjoint to T, 5) star is disjoint to trident, and 6) path is disjoint to trident. Most of those cases are similar to proving the disjoint for $c_1 = 3$ case, and some cases can be proven simultaneously.

First, let us show that star is disjoint to T, trident and path. Whichever tree that can be formed from the star unit has a black vertex of degree 4. In order for there to be an isomorphic tree to a tree that can be formed from a star, we need to have a black vertex of degree 4 in the path, trident, or T. However, there is no black vertex of degree 4 in the T unit, trident unit, or the path unit, and we cannot add more red vertices to the black vertex since it will violate our condition for $c_1 = 4$.

Second, T is disjoint to path or trident. The similar logic applies because T has a black vertex of degree 3, while neither path or trident does.

Lastly, path is disjoint to trident. The longest path that can be formed from the trident is 6, and the shortest path that can be formed from the T is 6. Thus, the only possibility for an isomorphic tree is a tree with diameter 6. However, the path has two red leaves when the diameter is 6, but in order to form a tree with diameter 6 from the trident, we will have black leaves in the path. Since the number of red vertices is fixed, these trees are not isomorphic.

These three cases are exhaustive. Every unit produces disjoint set of trees with $n \geq 9$ vertices where $c_1 = 4$. \square

While we won't discuss the details of partitioning each unit like we did for the $c_1 = 3$ case, we will discuss briefly an idea for counting the number of trees corresponding to each minimal unit. That is, since we can only add vertices to the red vertices in the minimal units, trees that are produced from each minimal unit are bijective to some kind of partitions or nonsymmetric partitions.

The star unit for $c_1 = 4$ is bijective to partitions of $n - 5$ into at most 4 parts, analogous to the star unit for the $c_1 = 3$ case. The trees from path unit for $c_1 = 4$ are bijective to nonsymmetric partitions of $n - 7$ into at most 4 parts (where we have (a, b, b, a)), also analogous to the path unit for the $c_1 = 3$ case. Trees from T and trident are a little more complicated in that we need to construct the partitions in a more careful manner. We can construct nonsymmetric partitions, but not up to reversal. From the labeling of red vertices of the T unit in **Figure 3.28**, a and d are symmetrical to each other, but b and c are not. In the trident unit in **Figure 3.29**, b , c , and d are symmetrical to each other while a is not. Despite some complications, partitioning of vertices is still helpful for counting the number of trees in each block.
(maybe elaborate more)

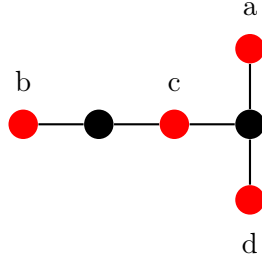


Figure 3.28: Labeling red vertices of T unit

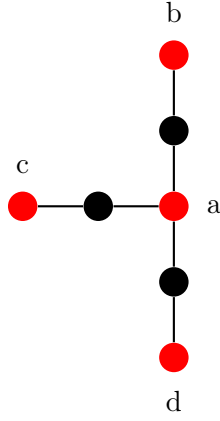


Figure 3.29: Labeling red vertices of trident unit

3.5 Generalization of minimal units

Having worked out examples with small c_1 values, we now generalize the minimal units. If $c_1 = k$, meaning one of the colors used was used exactly k times and the other color $c_2 = n - k$ times, trees will have the 2-CSF

$$X_T(2) = x_1^k x_2^{n-k} + x_1^{n-k} x_2^k.$$

It turns out that trees with $c_1 = 5$ have 9 minimal units, $c_1 = 6$ have 22 minimal units, and $c_1 = 7$ have 62 minimal units. Simply put, the numbers explode. Intuitively, this make sense since the minimal units for $c_1 = 5$ will be used to count trees

with 11 or more vertices, the minimal units for $c_1 = 6$ will be used to count trees with 13 or more vertices, and the minimal units for $c_1 = 7$ will be used to count trees with 15 or more vertices. From **Table 3.1**, while there are only 47 trees with 9 vertices, there are 235 trees with 11 vertices, 1301 trees with 13 vertices, and 7741 trees with 15 vertices. As the number of trees itself explodes, the number of minimal units that partition the trees would also explode.

With this in mind, we will now describe a process that can generate the minimal units for any value of k . In particular, we can generate minimal units for trees with the 2-CSF of

$$X_T(2) = x_1^k x_2^{n-k} + x_1^{n-k} x_2^k$$

when we have minimal units for $c_1 = 1$ through $c_1 = k - 1$ cases. Before we describe the procedure and demonstrate an example, we first state an intuitive remark for minimal units and a lemma that applies to bicoloring of trees in general.

Remark 1. *Any minimal unit cannot have black leaves; that is, every leaf must be red.*

This remark stems from an intuitive fact that if there is a black leaf in a tree, the tree can be produced from another minimal unit. Thus, the tree cannot be a minimal unit.

Lemma 2. *Trees with n vertices that can be properly bicolored with $c_1 = k$ have diameter at most $2k$.*

Proof. A tree with diameter $2k + 1$ contains a path with $2k + 2$ vertices that need to alternate in color. Thus, each color needs to be used $k + 1$ times, violating our condition that $c_1 = k$. □

This lemma is a generalization of **Proposition 8, 9, 10**. We now describe the process of producing minimal units for $c_1 = k$ using minimal units for $c_1 = 1, c_1 = 2, \dots$, and $c_1 = k - 1$.

Definition 23. *We can generate the minimal units for trees with n vertices and $c_1 = k$, given minimal units for trees with $c_1 = 1, c_1 = 2, \dots$, and $c_1 = k - 1$.*

1. *Start from the minimal unit of $c_1 = 1$. Convert all the red vertices in each minimal unit to black, and all the black vertices to red.*

2. *The above step produces black leaves. Add one red vertex to each black leaf so that there is no black leaf, as it is illegal by **Remark 1**.*

Note that the number of red vertices after this step is the (number of leaves + the number of black vertices) from the previous minimal unit.

3. *If the number of red vertices exceeds k following step 2, discard the minimal unit. Otherwise, if the number of red vertices is less than k , add more red vertices to the tree to have $c_1 = k$. Beware of isomorphism.*

4. *Repeat for $c_1 = 2$, and up to $c_1 = k - 1$.*

To assist readers' understanding of this process, we demonstrate an example to produce minimal units for $c_1 = 5$.

Example 6. *To generate minimal units for $c_1 = 5$, we must consider the minimal units for $c_1 = 1, c_1 = 2, c_1 = 3$, and $c_1 = 4$. For $c_1 = 1$ (**Figure 3.30**), you have a tree with one vertex. We convert the red vertex to a black vertex, and add five more red vertices to produce a star with 6 vertices (**Figure 3.31**).*

*Next, we consider the unit for the $c_1 = 2$ case, which is the spine of the caterpillar. We then swap the black and red vertices, which produces two black leaves. We have to add at least one red vertex to each black leaf, producing **Figure 3.32**.*



Figure 3.30: Minimal unit for $a = 1$

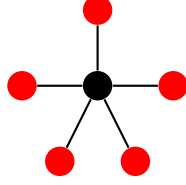


Figure 3.31: Star Unit for $c_1 = 5$



Figure 3.32: Middle step producing a minimal unit from spine

*Notice that we only have three red vertices in this middle step. Since we want to produce a minimal unit for $c_1 = 5$, we need to add two more red vertices to the black vertices. They can be put on the same vertex or one on each vertex, producing the following two minimal units depicted in **Figure 3.33** and **Figure 3.34**.*

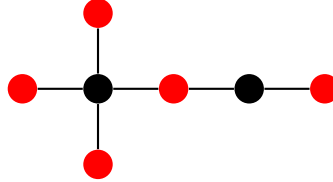


Figure 3.33: $(2, 0)$ unit from spine for $c_1 = 5$

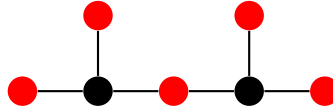


Figure 3.34: $(1, 1)$ unit from spine for $c_1 = 5$

*Now, we consider minimal units from $c_1 = 3$. There are two units: the path unit and the star unit. Lets first consider the path unit. We swap the red and black vertices, and add one red vertex to each black leaf. Then we get the tree in **Figure 3.35**.*



Figure 3.35: Middle step producing a minimal unit from path

We only have four red vertices in the middle step **Figure 3.35**. Thus, we need to add one more red vertex to one of the black vertices up to isomorphism, which yields the two minimal units in **Figure 3.36**

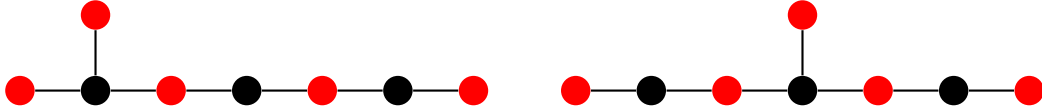


Figure 3.36: Two nonisomorphic minimal units for $c_1 = 5$

Then from the star unit of $c_1 = 3$, we can produce the following tree in **Figure 3.37** by swapping red and black vertices and adding red leaves to black leaves.

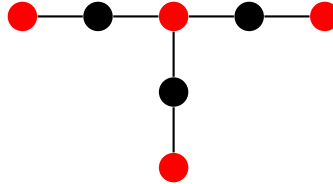


Figure 3.37: Middle step

Here, we need to add one more red vertex, and all black vertices are symmetric. This means that adding a red vertex to any of those black vertices will produce isomorphic units.

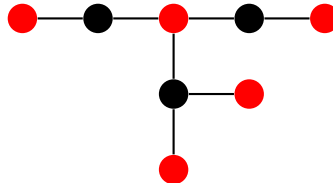


Figure 3.38: Extended F unit for $c_1 = 5$

Finally, we consider minimal units for $c_1 = 4$. Note that we cannot make a minimal unit for $c_1 = 5$ using the minimal unit trident because following steps 1 and 2, you have 6 red vertices, as demonstrated in **Figure 3.41**.

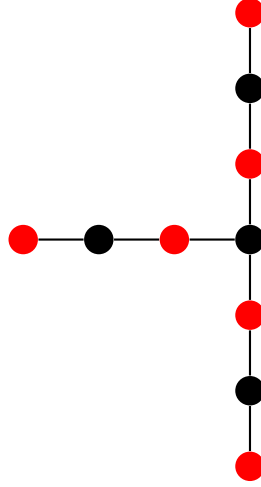


Figure 3.39: Violation of Minimal Unit

Thus, we should work with the other three minimal units for the $c_1 = 4$ case. The star has one black vertex and four red leaves, making the sum to $1 + 4 = 5$. For the T , we have two black vertices and three red leaves, again making the sum to $2 + 3 = 5$. Lastly, for the path, we have three black vertices and two red leaves, summing to $3 + 2 = 5$. This means that for each minimal unit for $c_1 = 4$ case, we can produce exactly one minimal unit for the $c_1 = 5$ case. The minimal units with four black vertices are shown in **Figure 3.40**, **3.41**, **3.42**.

This process produces 9 minimal units for the case $c_1 = 5$, as shown in **Figure 3.31**, **Figure 3.33**, **Figure 3.34**, two in **Figure 3.36**, **Figure 3.38**, **Figure 3.40**, **Figure 3.41**, and **Figure 3.42**. Following this example, readers may have a better understanding of how the minimal units are produced for bigger c_1 values and how the number of minimal units would increase for bigger c_1 values. Now, we should prove that the trees produced from such minimal units are disjoint.

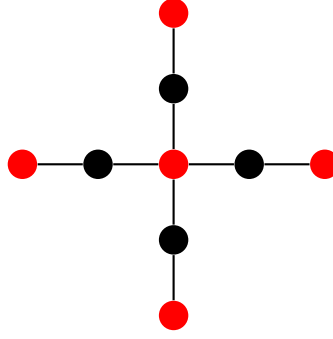


Figure 3.40: Cross

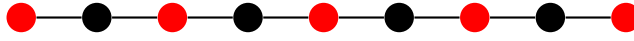


Figure 3.41: Path

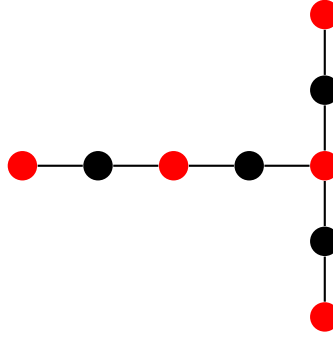


Figure 3.42: T

Theorem 8. *Minimal units for $c_1 = k$ case produces disjoint sets of trees with $n \geq 2k + 1$ vertices.*

Proof. The idea of this proof is by induction. From sections 3.1 through 3.4, we showed the base case. Assume that this theorem holds for some $c_1 = k - 1 \geq 4$. By

Definition 23, minimal units for $c_1 = k$ are produced from minimal units for $c_1 = 1$ through $c_1 = k - 1$, which are disjoint from one another. First, intuitively, minimal units that are produced from the minimal units in different blocks are disjoint since there are different number of black vertices. For instance, minimal units for $c_1 = 5$ produced from minimal units for $c_1 = 2$ are disjoint from minimal units for $c_1 = 5$

produced from minimal units for $c_1 = 3$. Then, minimal units that are produced from the same block are also disjoint. Say that we are producing minimal units for $c_1 = k$ using minimal units for $c_1 = m$, where $m < k$. Then we know that the minimal units for $c_1 = m$ are disjoint from one another by our assumption. Thus, all minimal units for $c_1 = k$ that can be produced from the process described in **Definition 23** are disjoint. \square

Following **Theorem 8**, we see that bicolourings of trees with n vertices where $c_1 = k$ are further partitioned by minimal units, as shown in **Figure 3.2**. To count each block, we can think of non-isomorphic ways to add black vertices to generate trees with n vertices. Thus, if a minimal unit has m vertices, then we need to add $n - m$ black vertices to produce a tree with n vertices.

Chapter 4

Distinguishing Trees with Chromatic Symmetric Functions

In this section, I will discuss briefly the basic properties of tricoloring of trees, that is, proper coloring of trees using exactly three colors. Instead of 2-CSFs, we will now use 3-CSF. Then, I will discuss how we may distinguish trees with two colors or three colors.

4.1 Tricoloring of trees

As mentioned in an earlier chapter, stars are an important basis for other trees. Thus, we first find a formula for the chromatic symmetric function of tricoloring of any star with $n \geq 3$ vertices. Denote by X_{st_n} the chromatic function for a star with n vertices. Note that $m_{x,y,z}$ denote a partition of $x + y + z$ into three parts, as defined by Richard Stanley. **Figure 4.1** describes a tricoloring of star with 6 vertices. In this case, the partition that corresponds to the tricoloring is $m_{4,1,1}$. Unlike in bicoloring, we cannot just swap the colors, which creates non-trivial coefficients for 3-chromatic symmetric

functions.

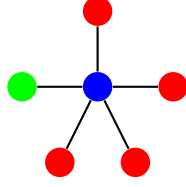


Figure 4.1: A tricoloring of star with 6 vertices

Proposition 11. *The 3-chromatic symmetric function for a star with $n \geq 5$ vertices is described by the following formula.*

$$X_{st_n}(3) = 2 \binom{n-1}{1} m_{n-2,1,1} + \binom{n-1}{2} m_{n-3,2,1} + \dots + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} m_{n-\lfloor \frac{n-1}{2} \rfloor, n-\lfloor \frac{n-1}{2} \rfloor, 1}$$

Proof. Note that $X_{st_1}(3) = e_1$ and $X_{st_2}(3) = e_2$. When $n = 3$, we have a path of length 3. Since we have three vertices, we must use each color exactly once. Since the colors can permute, we have

$$x_1 x_2 x_3 + x_1 x_3 x_2 + x_2 x_1 x_3 + x_2 x_3 x_1 + x_3 x_1 x_2 + x_3 x_2 x_1.$$

We can generalize the function as $6m_{1,1,1}$. Similarly, $X_{st_4}(3) = 6m_{2,1,1}$.

Consider $X_{st_n}(3)$ with $n \geq 5$. For each star with n vertices, we can construct a proper coloring by first fixing the color x_1 of the middle vertex (center), without loss of generality. The center vertex represents the last part in the partition $m_{\lambda_1, \lambda_2, \lambda_3}$.

Then we are left with $n-1$ vertices that must not be colored x_1 . Since $n-1$ vertices must be colored with the remaining two colors, we find partitions of $n-1$ into 2 parts. For each number n , the partitions can range from $(n-2, 1, 1)$ to $(\lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, 1)$.

Now we must determine the coefficients of each partition. Among $n-1$ vertices, we choose which vertices will be colored in x_2 and which vertices will be colored in

x_3 , without loss of generality. That is, we choose the second part in the partition λ_2 from $n - 1$. However, if we have 1 as the second part of the partition λ_2 , we obtain the same result by choosing the third part in the partition λ_3 from $n - 1$. Thus, we multiply the term by 2 when $\lambda_2 = 1$. This gives us the equation

$$X_{st_n}(3) = 2 \binom{n-1}{1} m_{n-2,1,1} + \binom{n-1}{2} m_{n-3,2,1} + \dots + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} m_{n-\lfloor \frac{n-1}{2} \rfloor, n-\lfloor \frac{n-1}{2} \rfloor, 1}.$$

□

Example 7. *A star with $n = 5$ vertices have the 3-chromatic symmetric function $6m_{2,2,1} + 8m_{3,1,1}$.*

A star with $n = 6$ vertices have the 3-chromatic symmetric function $10m_{4,1,1} + 10m_{3,2,1}$.

A star with $n = 7$ vertices have the 3-chromatic symmetric function $20m_{3,3,1} + 15m_{4,2,1} + 12m_{5,1,1}$.

Proposition 11 describes the 3-CSF of stars. To account for many more possibilities for tricoloring of trees, we will use the deletion-near-contraction relation (hereafter DNC algorithm) in [Or]. DNC algorithm utilizes stars as the basic unit of tree as it decomposes trees into stars to compute their chromatic symmetric functions.

Definition 24. [Or] *Let (G, w) be a weighted graph and e be a non-loop edge of G . Then we have*

$$X_{(G/e, w/e)} = X_{(G, w) \odot e} \setminus l_e - X_{(G, w) \odot e}.$$

*Moreover, the weighted chromatic symmetric function satisfies the **deletion near-contraction formula**:*

$$X_{(G, w)} = X_{(G \setminus e, w)} - X_{(G, w) \odot e} \setminus l_e + X_{(G, w) \odot e}.$$

Moreover, if G is simple, then

$$X_{(G,w)} = X_{(G \setminus e, w)} - X_{(G,w) \odot e^s \setminus l_e} + X_{((G,w) \odot e)^s}.$$

The DNC algorithm allows one to easily compute chromatic symmetric functions of trees with n vertices. Therefore, this algorithm is helpful for **Section 4.2**.

4.2 Distinguishing Trees

In this section, we explore how chromatic symmetric functions with restrictions on the number of colors used might distinguish trees. We computed chromatic symmetric functions on Stanley's monomial basis using sagemath.

Example 8. *The chromatic symmetric function of a star with 5 vertices is $120m_{1,1,1,1,1} + 36m_{2,1,1,1} + 6m_{2,2,1} + 8m_{3,1,1} + m_{4,1}$. When we restrict to exactly two colors, the 2-chromatic symmetric function is $m_{4,1}$. Similarly, the 3-chromatic symmetric function is $6m_{2,2,1} + 8m_{3,1,1}$.*

Intuitively, for a tree with n vertices, the largest number of colors that can be used is n . Therefore, when we consider k -chromatic symmetric functions, we will only consider trees with k or more vertices. The goal of this section is to see how many colors are needed to distinguish trees with n vertices.

4.2.1 2-Chromatic Symmetric Functions

When using two colors, the first time we see two non-isomorphic graphs sharing the same chromatic symmetric function is when $n = 5$, that is, for trees with five vertices. **Figure 4.2** depicts two trees with 5 vertices that have the same 2-chromatic symmetric function $m_{3,2}$.

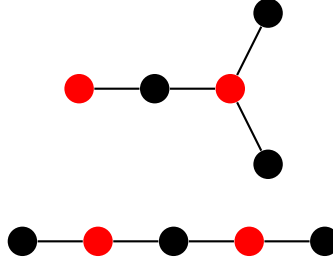


Figure 4.2: Bicoloring of Trees with 5 vertices

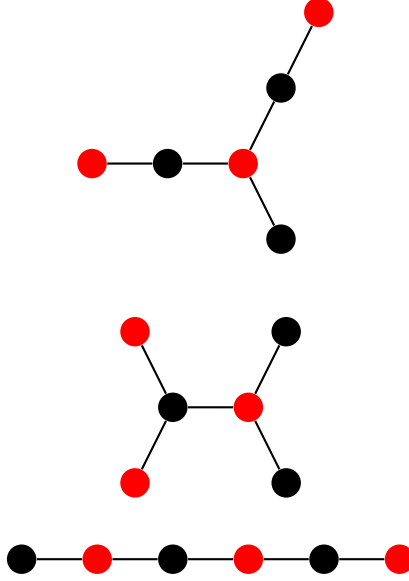


Figure 4.3: Bicoloring of Trees with 6 vertices

In fact, for $n \geq 5$, there are many trees sharing the same 2-chromatic symmetric functions. Among 6 trees with $n = 6$ vertices, there are two trees with 2-CSF $m_{4,2}$ and three with 2-CSF $2m_{3,3}$. **Figure 4.3** describes trees with $2m_{3,3}$. For trees with $n = 7$ vertices, there are 7 trees with 2-CSF $m_{4,3}$ and 3 trees with 2-CSF $m_{5,2}$. For trees with $n = 8$ vertices, there are 9 trees with 2-CSF $2m_{4,4}$, 10 trees with 2-CSF $m_{5,3}$, and 3 trees with 2-CSF $m_{6,2}$, consistent with **Table 3.1**. The next remark is another observation that leads to a difference between 2- and 3-chromatic symmetric functions.

Remark 2. *The only trees with 2-chromatic symmetric functions with a coefficient*

not 1 are equicolorable trees.

Equicolorable trees can have 2-chromatic symmetric functions with coefficient 2. For trees with $n = 2m$ vertices, if one of the colors used is used m times, the other color is also used m times. Therefore, swapping the two colors yields the same 2-chromatic symmetric function, making the coefficient 2.

Generally, 2-CSF cannot distinguish trees with n vertices. For example, there are 450 trees with 17 vertices where $c_1 = 4$. This means that there are 450 trees with the 2-chromatic symmetric function $m_{13,4}$.

Proposition 12. *2-chromatic symmetric functions cannot distinguish trees with n vertices that are not stars.*

This proposition follows from observing **Table 3.1** in Chapter 3.

4.2.2 Distinguishing Trees with Identical 2-Chromatic Symmetric Functions

In this section, we will discuss that it is possible to distinguish trees with identical 2-CSF using other k -CSFs. In order to approach the question, we outline an observation in **Lemma 3**.

Lemma 3. *Trees with n vertices are not distinguishable by n -chromatic symmetric functions or $(n - 1)$ -chromatic symmetric functions.*

Proof. First, trees with n vertices are not distinguishable by n -coloring chromatic symmetric functions because in every tree with n vertices, every vertex must be colored a different color in order for an n -coloring. The colors can permute, creating the coefficient of $n!$ for every tree with n vertices and arriving at the n -coloring chromatic symmetric function of $n!m_{1^n}$.

Such trees are also not distinguishable by $(n - 1)$ -coloring chromatic symmetric functions. In every tree with n vertices, two (non-adjacent) vertices can share the same color, and every other vertex must be colored differently. We can choose the color that can be used twice as well as the two vertices that are to be colored in this color in $\binom{n-1}{2}$ ways. Then, we have $n - 2$ colors that can permute. Combining the two, we have the formula

$$\binom{n-1}{2}(n-2)!$$

for the coefficient of $(n - 1)$ -coloring chromatic symmetric functions of every tree with n vertices. Checking this formula for $n = 5, 6, 7$, we get

$$\binom{4}{2}3! = 36,$$

$$\binom{5}{2}4! = 240,$$

and


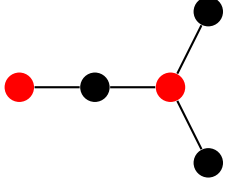
$$\binom{6}{2}5! = 1800,$$

consistent with the calculation on sagemath. □

Now that we are aware trees cannot be distinguished by their 2-chromatic symmetric functions, we turn our attention to k -chromatic symmetric functions of trees where $k \geq 3$. We will first consider bicolable trees with $c_1 = 2$, which are bicolable trees that have the 2-chromatic symmetric function of $m_{n-2,2}$. **Table 4.1** describes how trees with 5 vertices with identical 2-CSF may be distinguished by other k -coloring chromatic symmetric functions.

In the table, those trees with 5 vertices have the identical bicoloring, 4-coloring, and 5-coloring chromatic symmetric functions. However, they are distinguished by

Table 4.1: Distinguishing trees with $n = 5$ vertices

Tree with $n = 5$ vertices	Bicoloring	Tricoloring	4-coloring	5-coloring
	$m_{3,2}$	$12m_{2,2,1} + 2m_{3,1,1}$	$36m_{2,1,1,1}$	$120m_{1^5}$
	$m_{3,2}$	$10m_{2,2,1} + 4m_{3,1,1}$	$36m_{2,1,1,1}$	$120m_{1^5}$

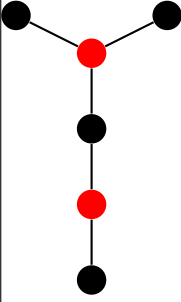
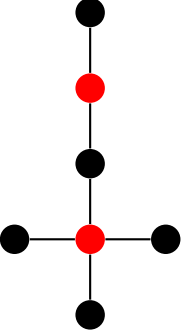

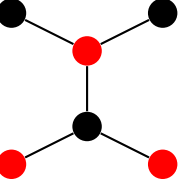
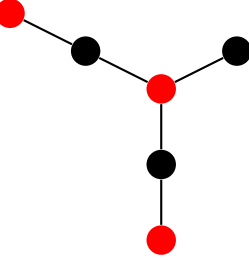
the coefficients of their tricoloring chromatic symmetric functions.

Now, we consider trees with 6 vertices. Excluding the star, there are two 2–chromatic symmetric functions for trees with 6 vertices, $m_{4,2}$ and $2m_{3,3}$, as described in **Table 4.2**. From the table, we again first consider trees with 2–chromatic symmetric function $m_{4,2}$. It seems that again, the trees are distinguished by 3–chromatic symmetric functions. Let us hypothesize that bicolorable trees with 2–CSF $m_{n-2,2}$ are distinguishable by their 3–CSF and prove it.

Theorem 9. *Trees with n vertices that have 2–CSF $m_{n-2,2}$ can be distinguished by their 3–CSF. Moreover, they are distinguished by the coefficient of the "most balanced" term in their 3–CSF.*

Before proving the theorem, we will define the "most balanced" term. The "most balanced" term of a 3-chromatic symmetric function (for a tree with n vertices) refers to the partition where we first divide n by 3 such that we have a partition $(\lfloor n/3 \rfloor, \lfloor n/3 \rfloor, \lfloor n/3 \rfloor)$. Then, we "fill" the remainder by adding to the partition starting from the first part while maintaining the balance. For instance, if we have $n = 7$, then we first have $(2, 2, 2)$ as the partition. However, since $2 + 2 + 2 = 6$, we add 1 to the first part, making our most balanced partition $(3, 2, 2)$. Similarly, if we have

Table 4.2: Distinguishing trees with $n = 6$ vertices

Tree with $n = 6$ vertices	Bicoloring	Tricoloring	4-coloring	5-coloring	6-coloring
	$m_{4,2}$	$30m_{2,2,2} + 9m_{3,2,1} + 2m_{4,1,1}$	$80m_{2,2,1,1} + 30m_{3,1,1,1}$	$240m_{2,1^4}$	$720m_{1^6}$
	$m_{4,2}$	$18m_{2,2,2} + 10m_{3,2,1} + 4m_{4,1,1}$	$72m_{2,2,1,1} + 42m_{3,1,1,1}$	$240m_{2,1^4}$	$720m_{1^6}$
	$2m_{3,3}$	$30m_{2,2,2} + 10m_{3,2,1}$	$84m_{2,2,1,1} + 24m_{3,1,1,1}$	$240m_{2,1^4}$	$720m_{1^6}$
	$2m_{3,3}$	$24m_{2,2,2} + 10m_{3,2,1} + 2m_{4,1,1}$	$76m_{2,2,1,1} + 36m_{3,1,1,1}$	$240m_{2,1^4}$	$720m_{1^6}$
	$2m_{3,3}$	$24m_{2,2,2} + 11m_{3,2,1}$	$80m_{2,2,1,1} + 30m_{3,1,1,1}$	$240m_{2,1^4}$	$720m_{1^6}$

$n = 8$, we also have $2, 2, 2$ as the partition, but we need to add 1 to the first two parts so that $3 + 3 + 2 = 8$. Thus, our most balanced partition for 8 is $(3, 3, 2)$.

Proof. Bicolorable trees with n vertices with 2-CSF of $m_{n-2,2}$ have a minimal unit, which is a spine with three vertices



Figure 4.4: Minimal Unit for $c_1 = 2$

We want to count the partition that corresponds to the most balanced term in the 3-chromatic symmetric function of the tree. Since we have three colors now, we can have two different ways to color **Figure 4.4**.

1. Case 1: All three vertices colored differently



2. Case 2: Two end vertices colored same



Among those two cases, Case 1 is of our interest as it shows why the coefficients are distinguishable. Every tree in the block $c_1 = 2$ has a one-to-one correspondence with a partition of $n - 3$ into at most two parts as we defined in **Chapter 3**. If we have a partition of $n - 3$ into one part, meaning the partition is $(n - 3, 0)$ with 0 as a padding, we cannot have any Case 1 balanced partition. Either the red vertex or the yellow vertex has degree $n - 2$, and the color can only be used once. For example, **Figure 4.5** represents a tree with 6 vertices with the corresponding partition $(3, 0)$. After fixing the three colors of the spine, we cannot add any more red vertices. This means that we cannot have the most balanced partition, which is $(2, 2, 2)$.

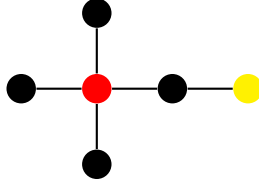


Figure 4.5: Cannot have a balanced partition

If we have a partition $(n-4, 1)$, then we have exactly one choice for the yellow/red degree $n-3$ vertex. It is the vertex that corresponds to the second part of this partition. Likewise, for each tree in the block $c_1 = 2$, the partition that corresponds to the tree determines the number of possible Case 1 most-balanced partitions.

Thus, the number of Case 1 most-balanced partitions depends on the partitions of $n-3$ into at most two parts, which decides the shape of trees in the $c_1 = 2$ block. Since every tree in this block corresponds to a different partition of $n-3$ into at most two parts, the number of Case 1 most-balanced partitions is different for each tree in the $c_1 = 2$ block. This makes the coefficients of the most balanced partition distinguishable.

□

Now that we have a distinguishing number for bicolable trees with 2-CSF $m_{n-2,2}$, the next step would be to expand and derive distinguishing numbers for bicolable trees with other 2-CSF. I will elaborate on those in **Chapter 5 Open questions**.

Chapter 5

Open Questions

In this thesis, we showed that we can classify trees with n vertices using bicoloring. We also showed that trees with n vertices are not distinguishable by 2-chromatic symmetric functions, except for stars with n vertices. Thus, in Chapter 4, we showed that the trees with 2-chromatic symmetric function $m_{n-2,2}$ can be distinguished by their 3-*CSF*. Continuing from the previous chapter, the main next step would be to continue and find ways to distinguish trees that belong to other blocks. That is, to find a distinguishing number for all trees with n vertices that are partitioned by bicolorings.

Moreover, there are some other unanswered questions. In Chapter 3, we showed that there are minimal units corresponding to every $c_1 = k$ value, or every 2-chromatic symmetric function $m_{n-k,k}$. As the number of minimal units explode from the case $c_1 = 7$, we were unable to investigate whether there is a significance to the sequence of the number of minimal units. We know that the 22 minimal units partition trees with 13 or more vertices where $c_1 = 6$, and 62 minimal units partition the trees with 15 or more vertices where $c_1 = 7$. Examining whether the number of minimal units for higher c_1 values will be interesting.

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