Optimal Betting for Fixpoint Solitaire

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May 29, 2012

Abstract

Let σ be a random permutation of $1 \ldots n$, and let X_i denote the location of the ith fixed point (or "fixpoint") of σ, where $X_i = 0$ if there are fewer than i fixed points. The expected value of X_1 is about $(1 - 2/e)n$ for large n. This represents the expected winnings of a card game in which a player wins an amount equal to the location of the first fixed point, but wins nothing if there is no fixed point. If we alter the payout system so that the player has the option to trade the payout i due to a fixed point at i for the expected value of the next fixed point, we find that this improves the expected winnings from $(1-2/e)n$ to $(1-\log 2)n$. The expected winnings for the second case are the same whether the player blindly uses a fixed cutoff point, or uses the theoretically optimal strategy, which continuously changes based on the exact set of elements of σ observed so far.

Introduction

The chance card game known as Fixpoint Solitaire derives its name because a player's goal is to find a "fixed point" in a deck of cards, which occurs if card k is in the kth position in the deck. The game requires no skill; a player's success depends entirely on what the order of the cards in the deck happens to be after the deck is shuffled. The rules of Fixpoint Solitaire are as follows. Consider an n-card deck, with no suits (it is just a stack of cards numbered 1 to n, and each number appears only once). The deck is shuffled so the cards are in a random order. A player takes cards one at a time off the top of the deck and turns them face up, and as he does so counts out loud "one, two, three..." up to n. If he ever turns up the card number that