On-Line Algorithms and Reverse Mathematics

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Abstract

In this thesis, we classify the reverse-mathematical strength of sequential problems. If we are given a problem P of the form

$$\forall X(\alpha(X) \to \exists Z\beta(X,Z))$$

then the corresponding *sequential problem*, SeqP, asserts the existence of infinitely many solutions to P:

$$\forall X\,(\forall n\alpha(X_n) \to \exists Z\forall n\beta(X_n,Z_n))$$

We will exactly characterize which sequential problems are equivalent to $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, or $\mathsf{ACA_0}$.

We say that a problem P is solvable by an *on-line algorithm* if P can be solved according to a two-player game, played by Alice and Bob, in which Bob has a winning strategy. Bob wins the game if Alice's sequence of plays $\langle a_0,\dots,a_k\rangle$ and Bob's sequence of responses $\langle b_0,\dots,b_k\rangle$ constitute a solution to P.

We show that SeqP is provable in $\mathsf{RCA_0}$ precisely when P is solvable by an on-line algorithm. Schmerl proved this result specifically for the graph coloring problem; we generalize Schmerl's result to any problem that is on-line solvable. We will then show that $\mathsf{WKL_0}$ suffices to prove SeqP precisely when P has a solvable closed kernel. This means that a solution exists, and each initial segment of this solution is a solution to the corresponding initial segment of the problem. (Certain bounding conditions are necessary as well.) If no such solution exists, then SeqP is equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2$; $\mathsf{RCA_0}$ alone suffices if only sequences of standard length are considered.

In Chapter 4 we analyze a variety of applications, classifying their sequential forms by reverse-mathematical strength.