

Orthogonal Modular Forms: An Application to a Conjecture of Birch, Algorithms and Computations

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Abstract

Orthogonal modular forms are algebraic modular forms arising from lattices in quadratic spaces. In this thesis, we define orthogonal modular forms, establish their basic properties, and then apply them to a case of ternary quadratic spaces to resolve a conjecture of Birch.

To this end we study the even Clifford map which is a faithful functor from the category of lattices in ternary quadratic spaces to the category of orders in quaternion algebras. We show that the even Clifford map preserves the Hecke-module structure between these two categories, and then resolve Birch's conjecture after applying the Jacquet-Langlands correspondence between quaternionic modular forms and Hilbert modular forms.

We also provide algorithms to compute Hecke operators for orthogonal modular forms and we bound their time complexity, demonstrating that such operators may be computed in polynomial time relative to $q = \#(\mathbb{Z}_F/\mathfrak{p})$ for a nonzero prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$. Moreover, we apply our algorithms to compute tables of Euler factors corresponding to some modular forms in $O(5)$ and $O(7)$, and make some observations and conjectures pertaining to these computations.