Sample questions for analysis preliminary exam

Problem 1. Let $A \subset \mathbb{R}$ be an open set and let $f\colon A \to \mathbb{R}$ be a function. Give three criteria (ϵ-δ, open sets, sequences) for f to be continuous on A. Show that two of these definitions are equivalent.

Problem 2. Prove that for all $x > 0$ we have the inequality

$$\sin x > x - \frac{x^3}{6}.$$

Problem 3. Show that if the uniformly continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ for $n \geq 1$ converge uniformly to $f\colon \mathbb{R} \to \mathbb{R}$, then f is uniformly continuous.

Problem 4. Let (X, d) be a compact metric space and $f\colon X \to X$ be a continuous function such that if $x \neq y$, then $d(f(x), f(y)) < d(x, y)$. Show that f has a unique fixed point.

Problem 5. Let U be a connected, open subset of $\mathbb{R}^n$. Suppose $f\colon U \to \mathbb{R}$ is a function that is differentiable on U and that all partial derivatives $\dfrac{\partial f}{\partial x_i}(p) = 0$ vanish for all $p \in U$. Prove that f is constant.

Problem 6. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a monotone, decreasing function defined on the positive real numbers with

$$\int_0^\infty f(x)\,\mathrm{d}x < \infty.$$

Show that

$$\lim_{x \to \infty} x f(x) = 0.$$

Problem 7. Suppose that X and Y are topological spaces with Y compact, and give $X \times Y$ the product topology. Show that the projection map $\pi\colon X \times Y \to X$ is a closed map.

Problem 8. Give an example of a Hausdoff topological space X and an equivalence relation $\sim$ on X so that the topological space $Y = X/\sim$ is not Hausdorff.

Problem 9. Prove or disprove: the set $\mathbb{Q}$ of rational numbers is the intersection of a countable family of open subsets of $\mathbb{R}$.