Math 116 : Homework 2—corrected version

due Thurs Feb 2; I suggest you spread out the work

Tips: Read comments to HW1, your own feedback, and fix code from qu. 3 since you'll use it now. Try to put up your code online, in whatever shape or form. Please label your plots.

1. Use a double-layer potential integral equation to solve the interior Dirchlet BVP,

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ u &= f \qquad \text{on } \partial\Omega, \end{aligned}$$

on a rotated version of the 'trefoil' domain from last time described in polars by $r(\theta) = 1 + 0.3\cos(3\theta + \pi/10)$. Use the Nyström method with N boundary points; HW1 gave you almost all the tools for this. Make your code accept a general function f given as a vector of values at the boundary points. Don't forget the limit of the kernel on the diagonal we derived in class.

   (a) A good way to test accuracy is to arrange the solution to be known, but unrelated to the domain shape, *e.g.* the harmonic function $u = x_1 x_2$ where $(x_1, x_2) = x \in \mathbb{R}^2$. (Note: we rotated Ω above to break any reflection symmetry). Let f take the values of this function on $\partial\Omega$. Call $\hat{u}_N$ your resulting numerical interior solution using N points. Make 3D plots of the function $\hat{u}_{50}$ and the error function $\hat{u}_{50} - u$, making sure to use high resolution, *i.e.* n by n points with $n \geq 100$. [Bonus: in your plot exclude points outside Ω, by dividing them by 0 to create an `Inf` which isn't plotted].

   (b) You'll notice that height is hard to judge on such plots. 2D graphs are better in this respect. Plot a *slice* through the error function, that is, plot as a function of x_1, at fixed $x_2 = 0.5$.

   (c) Plot convergence of error measured at the single point $(0.25, 0.25)$, vs N. You may choose a few N's by hand, or, better, loop over N. How do you describe the convergence of the BIE solution at the single point? Compare this against convergence you would expect for our quadrature rule (are integrands C^2, analytic, etc?).

   (d) Invent an $r(\theta)$ with a single corner (do not let $dr/d\theta$ get big anywhere otherwise quadrature points will be too far apart there). Solve the same problem as above in this new domain. How is convergence now?

2. Use BIE to solve the exterior Helmholtz scattering problem in 2D, for the above domain with Dirichlet boundary conditions, and incoming wave $u^i(x) = e^{ik\hat{n}\cdot x}$ with $\hat{n} = (\cos 0.2, \sin 0.2)$.

   (a) Use a double-layer potential to represent the scattered field. Choose $k = 10$ and produce a labeled color density plot of $u(x)$ similar to that on the top of our course webpage (however notice I sneakily zeroed the interior field—in truth it is nonzero, but not physically relevant).

   (b) Using your resulting layer density, plot the *far field pattern* $u_\infty(\hat{x})$ as a function of angle $\hat{x} = \theta \in [0, 2\pi]$. Redo it at a larger N value to check you've reached convergence to a decent accuracy.

   (c) Plot the condition number of your discretized matrix $(\tilde{K} + \frac{1}{2}I)$ as a function of wavenumber k in the range 0 to 2. Note this doesn't require solving any linear system or evaluating any fields. What do you notice around $k = 1.5$? Find the problem k value as accurately as you can. This is the *first interior Neumann resonance*.