Complex Analysis Assignment 1

Sometime, 2008

1. Let $\gamma : [a,b] \to \mathbb{C}$ be a path and $\Omega = \mathbb{C} \setminus \gamma^*$. In class we proved that the function $Ind_\gamma : \Omega \to \mathbb{C}$ defined by

$$z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{w-z} dw$$

is integer-valued and concluded that it is constant on the conected components of Ω. To do this, however, requires that Ind_γ be continuous. Prove this by appealing directly to the definition of continuity.

2. Suppose that $\gamma : [a,b] \to \mathbb{C}$ is a closed path with $0 \notin \gamma^*$. Prove that $\int_\gamma z^n dz = 0$ for $n = -2, -3, \ldots$.

   Hint: Find antiderivatives.

3. Let $\Omega \subseteq \mathbb{R}^2$ be an open subset of the plane. Recall from multivariable calculus that the line integral of a vector field $\vec{F} : \Omega \to \mathbb{R}^2$ along a curve $\gamma : [a,b] \to \Omega$ is defined to be

$$\int_\gamma F_1 dx + F_2 dy := \int_a^b \langle \vec{F}(\gamma(t), \gamma'(t) \rangle dt = \int_a^b (F_1(\gamma(t))\gamma_1'(t) + F_2(\gamma(t))\gamma_2'(t)) dt.$$

   Let γ be the positively oriented circle of radius 1 centered at 0.

   (a) Let $\vec{R}$ be the the vector field on $\mathbb{R}^2 \setminus \{0\}$ given by $\vec{R}(x,y) = (\frac{x}{r^2}, \frac{y}{r^2})$ where r is the radial distance fom the origin. Prove that the integral of $\vec{R}$ along γ is zero.

   (b) Let $\vec{C}$ be the vector field on $\mathbb{R}^2 \setminus \{0\}$ given by $\vec{C}(x,y) = (\frac{-y}{r^2}, \frac{x}{r^2})$. Prove that the integral of $\vec{C}$ along γ is 2π.

   (c) The real and imaginary parts of $Ind_\gamma(0)$ can be expressed in terms of line integrals of vector fields along γ. What are these vector fields?

   Hint: Write $\int_\gamma \frac{1}{z} dz$ as $\int_0^1 \frac{1}{\gamma(t)} \gamma'(t) dt$ and break down into real and imaginary parts.

   (d) Explain geometrically why $Ind_\gamma(0)$ is nonzero.